Chasing a Conjecture

Inside the Mind of a Mathematician

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juggernaut

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To Aai and Baba

तुका तुकासी तुकला । तुका तुकाहुनि निराळा ॥
तुकीं तुकला तुका । विश्व भरोनि उरला लोकां ॥

– TUKARAM
seventeenth-century poet–saint

Let us grant that the pursuit of mathematics is a divine madness of the human spirit, a refuge from the goading urgency of contingent happenings.

– ALFRED NORTH WHITEHEAD
1861–1947, mathematician and philosopher

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Introduction

Symmetry, Symmetry, Symmetry

I am a number theorist, which means I think about questions related to what seem like the simplest of all things: the natural numbers 1, 2, 3, ... To say that something is straightforward, one says that it is as easy as one, two, three. And yet, the emergence of numbers is considered a major breakthrough in human civilization. Numbers allowed us to understand the world in powerful new ways. Numbers abstract and make precise and quantify our notions of size and numerousness. The number two is an abstraction and captures what is common between all pairs of things: two apples, two chairs, two horses.

Number theorists look at 1, 2, 3, ... for their own sake, and are tantalized by simple-sounding questions about them, whose answers often lie very deep. Some of the hardest questions in mathematics are about the natural numbers. A mathematician can obsess over these questions for years. Collectively, they have driven mathematical work over hundreds of years. Once lured into thinking about an innocent-looking question, one is hooked and can't let go of it – the more one understands its difficulty, the less one is able to move away from it. A wonderful example of such a question is one posed by Pierre de Fermat in the seventeenth century.

According to the famous Pythagoras theorem, in a right-angled triangle, the square of the length of the hypotenuse (the side opposite the right angle), is equal to the sum of the squares of the lengths of the other two sides. Mathematically, this is expressed as $a^2 + b^2 = c^2$. There are many solutions to the equation $a^2 + b^2 = c^2$ where *a, b, c* are natural numbers; $3^2 + 4^2 = 5^2$ or $5^2 + 12^2 = 13^2$ are just two of the infinite number of examples.

In 1637, Fermat mused if this could happen for powers of numbers higher than just their squares. Can a cube be the sum of two cubes, that is can $a^3 + b^3 = c^3$? Can a fourth power be the sum of two fourth powers? He made the assertion that this could never happen for powers bigger than 2, and further said that he could prove his assertion. He jotted this in a margin of the book that he was reading – Diophantus's *Arithmetica* – and noted that the margin was too small to contain his marvellous demonstration.

The assertion became notorious, as no one could prove it for the next three hundred and fifty years. It came to be known as Fermat's Last Theorem (FLT). In the 1960s, a ten-year-old boy stumbled upon the problem in a book called *The Last Problem* in a public library of his home town of Cambridge in England, and became fascinated by it. The boy was Andrew Wiles who went on to become a celebrated number theorist. FLT became an actual theorem only in 1994 when Wiles proved it after a herculean effort of seven years.

A theorem is like a statement of fact: the Pythagoras theorem, which states a relation between the side lengths of a right-angled triangle, is an example. In mathematics, the statement of a theorem has to be followed by a proof. Once proved, a theorem might become the foundation of a theory, or be able to solve problems by simply being invoked. To quote a theorem is almost like casting a spell. The tremendous effort and ingenuity expended in proving a theorem is compacted into its great power that can be activated by simply quoting it.

Fermat's problem is a mere riddle, by itself of no larger significance.[1] Yet, amazingly enough, Fermat's problem acted as a muse, leading to the discovery of important mathematical theories through the ensuing centuries. Many mathematicians engaged with it, developing methods that could prove it for powers that went up to several million. This was not the same, however, as proving it conclusively.[2]

Wiles's monumental proof of FLT is more than a hundred pages long. In his proof, Wiles is engaged in a much bigger enterprise than proving FLT. He states it as a theorem in the introduction of the paper. The rest of the paper is devoted to proving something called the 'elliptic symmetry conjecture'. If a theorem is a proven fact, a conjecture is like an unfulfilled fantasy in the world of mathematics in which the most fantastic things happen routinely. The best conjectures are more than just guesses, often having passed many plausibility tests. A conjecture reflects the sunny yet cautious optimism of the mathematicians who propose it: the conjecture would have to be true if we were living in the best of all possible worlds. A conjecture becoming a theorem is like a fantasy coming true, and takes the hard work and brilliant ideas of mathematicians who are consumed by their efforts to find its proof.

Mathematical developments in the 1980s showed that Fermat's problem is a wholly unexpected and fortuitous consequence of the powerful elliptic symmetry conjecture, made in the 1950s, that is at the heart of modern number theory. For the first time, there was a compelling mathematical reason to demonstrate that Fermat was right: the elliptic symmetry conjecture was too important for it to fail. Wiles was stimulated to work on his childhood dream of solving Fermat's question once it was found to be a quick corollary of the elliptic symmetry conjecture.

Wiles's proof illustrates an important mathematical idea: the power of abstraction. The development of mathematical ideas starts with a very concrete problem, such as FLT, as a provocation. However, if the problem is deep, it cannot be resolved simply by brute force computation. Its elusiveness can lead mathematicians on a long journey of developing ideas that seemingly take them further and further away from the problem. Mathematicians become increasingly interested in the ideas for themselves. In the case of FLT, the abstract theories it set in motion solved it almost as an afterthought.

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Wiles's marvellous proof of Fermat is very indirect. He uses ideas related to studying the 'symmetries' of solutions of equations. The use of symmetry in such questions goes back to the work of the extraordinary nineteenth-century French mathematician Évariste Galois, who died in a duel when he was just twenty but still managed to produce work that would revolutionize mathematics. Galois used abstract ideas of symmetry to answer a question about solutions of polynomial equations that had been around for hundreds of years.

In school, we learn the formula used since antiquity for solving one kind of polynomial equation: the quadratic, or second degree, equation. The formula involves taking square roots.[3] Mathematicians sought similar formulas for solutions of polynomial equations of higher degrees. In the sixteenth and seventeenth centuries, similar formulas using only radicals (extracting square roots, cube roots, and so on) were found for solutions of polynomial equations of third and fourth degree, but these solutions were more complicated than the one formulated for the quadratic equations.[4]

Despite strenuous efforts, no such formulas, using only radicals, could be found for solutions of polynomial equations of fifth degree and higher. Galois hit upon the idea of studying the symmetries of solutions of polynomial equations rather than the solutions themselves.[5] Before his tragic death, Galois wrote letters and manuscripts from a prison, summarizing his breakthrough ideas. One of them has the recurring sentence *Je n'ai pas le temps* (I do not have time) as an anguished refrain. Galois' work shows that there is a compelling reason that no one for centuries had been able to find clean-cut formulas for roots of polynomials of degree higher than 4: it was proven to be an impossible task.[6]

As a simple illustration of the idea of Galois symmetries, consider the quadratic equation $X^2 - 5 = 0$ with its solutions $\sqrt{5}$ and $-\sqrt{5}$: their Galois symmetry swaps them by changing the signs. This symmetry is analogous to the famous bilateral symmetry of the Taj Mahal. If we imagine a huge vertical mirror going through the middle of the Taj Mahal, the reflection of either side in the mirror looks the same as the part on the other side of the mirror. The Galois symmetry of the equation $X^2 - 5 = 0$ and the bilateral symmetry of the Taj Mahal both have the property that if you repeat them twice, you are back to square one.

I am writing this introduction sitting in a cafe, and I can see in its windows the reflection of a lovely vase with vivid red blossoms. Part of its beauty for me lies in the fact that it is balanced and symmetrical. Just as numbers abstract the intuitive ideas we have about quantity, mathematical ideas about symmetry abstract our intuitions about beauty and balance. Galois' ideas show that the symmetries of the solutions of a polynomial equation, whether or not they capture the equation's beauty, have a functional bearing on its solutions, revealing something vitally important about their nature.

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Mathematical ideas can be likened to viruses that infect mathematicians who end up becoming their carriers and transmitters. Once the idea that would go on to be known as 'Galois symmetry' infected its creator, its highly contagious nature ensured its spread across the mathematical community.

Wiles's proof is in the lineage of Galois' work on the impossibility of solving a general polynomial equation of a degree higher than 4 using only radicals. His proof is by contradiction: Wiles shows (via his work on the elliptic symmetry conjecture) that a counterexample to Fermat's assertion gives rise to a forbidden Galois symmetry. To give the tiniest hint about what this involves, Wiles rules out the Galois symmetry arising from a solution to Fermat's equation by relating it to a completely different kind of symmetry, namely a Ramanujan symmetry.[7] The latter type of symmetry was used to understand the genius mathematician Srinivasa Ramanujan's deep and subtle observations about patterns in a sequence of numbers that he described in a paper he wrote in 1916.

The only known way to rule out the Galois symmetry arising from a solution of the Fermat equation is by relating it to the far more tangible world of Ramanujan symmetries. Galois' work ended the age-old quest to find formulas using only simple-minded operations (like taking radicals) for solutions of polynomial equations but it started the still-ongoing quest to understand the complexity of their solutions, and their Galois symmetries, using various other types of symmetries. This is typical in mathematics: answering a compelling question gives rise to further fascinating questions, and so on ad infinitum. No answer is the last word on a subject.

When we talk about Wiles's work on symmetry, we are inevitably led to talking about a conjecture of Jean-Pierre Serre. Serre is a celebrated French mathematician, who made a powerful conjecture in the 1970s and 1980s about the connection between Galois and Ramanujan symmetries. It is like a beautiful metaphor which brings together two very different ideas. To make an analogy with physics, one could think of the two symmetries as similar in a way to electricity and magnetism. In the nineteenth century, Maxwell proposed his equations that tied electricity and magnetism together, making them two aspects of the electromagnetic field. This gave rise to new physics that emphasized fields rather than mechanical forces and led to the uncovering of the electromagnetic spectrum, with many applications (such as radio, TV, wireless internet) that make our modern way of living possible. In an analogous way, Serre's conjecture unifies the Galois and Ramanujan symmetries and has powerful applications within mathematics: a proof of the elliptic symmetry conjecture and FLT is part of its potent force field.

In mathematics, analogies between concepts that are very different play a vital role. André Weil, one of the giants of mathematics in the twentieth century, wrote about the importance of such analogies in a letter he sent to his philosopher sister Simone Weil from a prison in 1940. He had landed in prison because of dodging the draft to fight as a French soldier in World War II. In his letter, Weil compared the role of analogies in mathematics to the Rosetta Stone[8] which had the same text inscribed on it in Demotic, Ancient Greek and Egyptian hieroglyphics. The stone's discovery led to the deciphering of the Egyptian hieroglyphics. Serre's conjecture is like a Rosetta Stone that allows one to translate back and forth between Galois symmetries and Ramanujan symmetries. Before the conjecture was proved, mathematicians had not been able to invoke it unconditionally.

Wiles's work in 1994 showed that very particular types of Galois symmetries are the same as Ramanujan symmetries: this was enough to prove FLT but left most of Serre's conjecture untouched.[9] Despite this, there was hope that as his work gave a new method to relate Galois and Ramanujan symmetries, it might help with attacking Serre's conjecture.

My enduring love of symmetry, since the time I first encountered Galois theory as an undergraduate at Cambridge in the late 1980s, made the challenge of proving Serre's conjecture irresistible for me. Mathematical research is like a relay race and I wanted to pick up the baton after Wiles's astonishing work and run as much of the stretch as I could to the solution of Serre's conjecture. When I began, I had no idea how long this could take, or how far I or anyone could get. Many mathematicians all over the world had also picked up the baton and wanted to sprint their way to the proof of Serre's conjecture. The quest for its proof was that of a whole community of mathematicians spread all across the globe.

Pure mathematics might appear esoteric, but it can get intensely competitive. When the stakes are high and people are converging towards a breakthrough, many emotions are at play: envy, ambition, one-upmanship. After all, one is playing for a smidgen of immortality! Politics and gamesmanship can abound belying the 'pure' in pure mathematics. But it is these impurities that create the compelling alloy of the subject as practised at the frontiers of research.

Proving Serre's conjecture became my moonshot soon after I finished my PhD thesis in 1995. It took me more than a decade of work before my French collaborator Jean-Pierre Wintenberger and I could settle the conjecture in 2008. The mathematical plot of this book is about mathematicians who got infected by the 'virus' of Galois and Ramanujan symmetries, and as a result became obsessed, or afflicted, with them. Magnificent work throughout the twentieth century linked these two very different types of symmetries, bringing them closer and closer together. The proof of Serre's conjecture is a milestone in showing that these symmetries, in spite of their totally different morphology, astonishingly enough have essentially the same DNA. This book is mainly about the human story of this mathematical achievement.

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The interaction between mathematics and the person thinking about it unfolds almost like a dance. The subject enchants by its elegant questions and its powerful ideas. Even as it invites one in, it resists by its difficulty, its particular ways of argumentation that need to be learned. One cannot absorb the subject passively. There is little instant gratification on offer. We see in this book a researcher tussling with his limitations and inadequacy. I often thought of giving up during my struggles as a graduate student, or even later when I was making little progress in my research, when I felt overwhelmed by the difficulties of the subject.

The subject initially resists understanding, as if demanding that the student put in the necessary effort to grasp its core ideas and vital essence. A student experiences frustration and joy in battling and overcoming this resistance, and wants to learn further, go deeper, be all in, because there is always more to contend with, understand and then delight in. The work done in overcoming the resistance tests resilience and builds reserves of strength and experience to a point where a student can begin to add to the subject in modest ways, ask questions of it that lead to new insights and new connections. If you are lucky, the dance may also be able to draw out strengths within that you were unaware of at the beginning. Almost everything that I learnt or had engaged with seriously in mathematics got used in our proof of Serre's conjecture. My thesis work on manipulating Ramanujan symmetries gave me key intuitions that led to breakthroughs in the work years later on Serre's conjecture. (Many mathematicians can attest to this experience of feeling that mathematics cannibalizes all of yourself that you offer to it!)

This dance is not unique to mathematics, and probably occurs across all creative pursuits in challenging fields that require significant apprenticeship before one can begin original work. Indeed, mathematical research is regarded in this book as a particular kind of creative activity, cognate to composing music, designing a piece of public architecture or writing a novel. Similar, but with its own marked differences and rugged individuality. I hope this narrative-based account of creativity in mathematics will resonate with the readers interested, more generally, in the mysterious ways by which ideas arise in the mind, solving problems that seem impenetrable when one first thinks about them.

Thinking about mathematics is an important part of a mathematician's life, but it is far from all of it. Mathematical work is done in the midst of all that happens in one's life: what a person does when not thinking about mathematics affects decisively the mathematics they do. In a related vein, the nineteenth-century French mathematician Charles Hermite said it is the person and not the method that solves a mathematical problem. The open-ended nature of mathematical research leads to a freedom that provides scope for the expression of personal qualities, which can result in one seeing a problem in a way that is new. Contrary to the general perception that mathematics is a highly technical and hermetic subject that only certain types of people can study, in practice people from different backgrounds, and a breadth of sensibilities and talents, can make a great impact on the subject. But perhaps because of the way mathematics is viewed and its reputation as a Procrustean subject, there is a notable lack of diversity, across many aspects, in the mathematical community. Hermite's actual quote, 'It is the man, and not the method, who solves a problem' reflects the historic male dominance in the field at his time, one that sadly continues to this day. There is typically just a scattering of women mathematicians present at the conferences I attend. This is to the great detriment of the subject. A greater diversity in its practitioners would not only enrich the social context within which it is practised but also enhance and transform the theoretical core of the subject.

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One's mathematical creativity can draw sustenance from family and social ties. In my journey to the proof of Serre's conjecture, the support of my family was crucial. One also draws from one's other interests – poetry, music, art, or sports, to name a few. What this book also tries to bring alive is that mathematics is increasingly taking the form of a social endeavour in a world community of mathematicians. With the improved communications of the later part of the twentieth century, mathematics has become a far more collaborative enterprise than in the past; however, it goes without saying that it still requires a lot of individual effort. The popular conception of a mathematical scholar working in splendid isolation in a garret and discovering mathematical truths – while it might apply to a very few select celebrated figures, like Sir Isaac Newton or Carl Friedrich Gauss, or to the great Indian mathematicians Srinivasa Ramanujan or Harish-Chandra – is simply not the common experience of most mathematicians today.

I benefited from learning from the work of many mathematicians. I was inspired by direct contact with, and the personalities of, some of them. I was often lucky to be in the right place at the right time. Wiles's methods becoming available right around the time I finished my PhD was great timing for me. Furthermore, the proof of Serre's conjecture would not have been possible without the extensive work of many people that led to crucial developments all through the 1980s and 1990s.

The book may sometimes give the impression that mathematicians frequently sojourn in exotic locations in the cause of doing research; visits to places like Sapporo and Paris and Strasbourg feature prominently in my mathematical journey. While such trips are undeniably enjoyable as a way to experience different parts of the world, they can sometimes catalyse a breakthrough just by the alchemy of being in the same place as a colleague and asking a question or sharing an idea that would have never made it into an email, or just by the neurons being jostled by travel and the stimulation and novelty of new surroundings.

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The book does not try to teach its readers mathematics, at least not in any formal sense, but it does aim to make it vividly present to them. I have tried to explain, in plain English, the development of ideas from Gauss's eighteenth-century quadratic reciprocity law to Ramanujan's prophetic 1916 paper to the formulation of Serre's conjecture in the 1970s and 1980s, culminating in my proof of it with Wintenberger in the first decade of the twenty-first century.

In my initial drafts of this book, there was little mathematics. My intention was to focus on the creative endeavour, immerse the reader in the atmosphere of mathematical research, give a sense of